Crossover exponent in O(N) ϕ^4 theory at $O(1/N^2)$

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The critical exponent ϕ_c, derived from the anomalous dimension of the bilinear operator responsible for crossover behavior in O(N) ϕ^4 theory, is calculated at $O(1/N^2)$ in a large N expansion in arbitrary space-time dimension $d=4-2\epsilon$. Its ϵ expansion agrees with the known $O(\epsilon^4)$ perturbative expansion and information on the structure of the five-loop exponent is provided. Estimates of ϕ_c and the related crossover exponents β_c and γ_c, using Padé-Borel resummation, are provided for a range of N in three dimensions.

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Renormalization group techniques have been widely used to study the critical properties of the scalar quantum field theories underlying a variety of condensed matter systems. For instance, Wilson and Fisher [1–3] introduced the technique of extracting numerical estimates of critical exponents from the evaluation of the perturbatively calculated renormalization group functions at several loop orders in ϕ^4 theories. Subsequently various authors have developed this method to very high loop orders either in fixed, (three), space-time dimensions or in $d=4-2\epsilon$ dimensions. The results for various renormalization group functions at, respectively, six and five loops, which represents the highest orders computed, are given in Refs. [4,5]. In the latter case the results have been extrapolated to three dimensions using resummation techniques [5–7]. These exponents derived by the renormalization group method are competitive with other approaches such as the high-temperature series expansion and Monte Carlo results and are in good agreement with experiment. A recent and comprehensive review of the application of the renormalization group in this area is given in Ref. [8]. Recently the critical exponent corresponding to crossover behavior in O(N) ϕ^4 theory has been calculated to a new degree of accuracy in fixed dimension [9] where the relevant Feynman diagrams were calculated to *six* loops. One motivation for that study rests in the realization that the $N=5$ theory of superconductivity has been observed in nature [10]. Prior to this the same exponent had been computed to four loops in $\overline{\rm MS}$ in $d=4-2\epsilon$ perturbation theory in Ref. [11], which built on the lower loop calculations of Refs. [3,12]. The resulting numerical estimate for that crossover exponent was in agreement with the high-temperature series of Ref. [13]. One other field theoretic technique that is used in estimating critical exponents is the large N method where the exponents are computed order by order in powers of $1/N$. Indeed exploiting the conformal properties of the d-dimensional Wilson-Fisher fixed point the technique has successfully produced the critical exponent η at $O(1/N^3)$ in d dimensions [14] through use of the conformal bootstrap program. Moreover, this method had developed out of the earlier d-dimensional critical point technique of Refs. [15,16] that was based on analyzing Schwinger-Dyson equations at criticality in large N. In essence the method efficiently reproduces the bubble summation that is the main property of $1/N$ expansions but more importantly goes beyond the leading order that the conventional bubble sum calculations fail to handle easily. Moreover, since the exponents are expressed as a function of d they can be expanded in powers of ϵ and the coefficients compared with those of the same exponents computed in conventional perturbation theory. Due to the critical renormalization group the coefficients must be in agreement. Therefore, the information contained in the large N exponents can be exploited, for instance, to gain insight into the large order structure of the renormalization group functions at several orders in $1/N$. Given the recent interest in the crossover exponent we will focus in this paper on its evaluation at a *new* order in $1/N$ in O(N) ϕ^4 theory. Previously the exponent had been calculated at $O(1/N)$ in d-dimensions in Refs. [17,18]. To achieve this we follow the extension of the large N fixed point Schwinger-Dyson approach of Refs. [15,16], to the computation of the anomalous dimensions of composite operators [19].

We recall the essential points of our calculation. The crossover exponent we are mainly interested in, ϕ_c, is computed from the anomalous dimension of the bilinear traceless symmetric tensor,

$$T^{ab}=\phi^a\phi^b-(\delta^{ab}/N)\phi^c\phi^c, \tag{1}$$

where ϕ^a is the field of the O(N) ϕ^4 theory and $1\leq a\leq N$, through the scaling law

$$\phi_c=(2-\eta_c)\nu. \tag{2}$$

This composite operator is relevant for a variety of critical phenomena [9,13,20,21]. The exponent ν has been computed in d dimensions at $O(1/N^2)$ in Ref. [16]. To clarify with other work [9] the exponent η_c is related to two other exponents η and $\eta_{\mathcal{O}}$ by

$$\eta_c=\eta+\eta_{\mathcal{O}}, \tag{3}$$

where η is the anomalous dimension of the field ϕ^a and has been computed at $O(1/N^3)$ in Ref. [14]. The remaining exponent $\eta_{\mathcal{O}}$ is the anomalous dimension of the bare composite operator T^{ab} itself. We have chosen to express the relation for ϕ_c in this way since in a gauge theory the combination η_c would be independent of a covariant gauge parameter although the analogous η and $\eta_{\mathcal{O}}$ would each depend on the choice of gauge. In the large N critical point method of Ref.

[19] the exponent $\eta_{\mathcal{O}}$ is extracted by inserting the operator T^{ab} in a two point Green's function and extracting the residue of the simple pole with respect to the large N regularization in a well defined fashion according to Refs. [15,16]. The residues of the simple pole of each Feynman diagram are then combined to obtain $\eta_{\mathcal{O}}$. Before recalling how this regularization is introduced we note that the Lagrangian used in the large N technique is

$$L = \tfrac{1}{2}\,\partial^{\mu}\phi^{a}\partial_{\mu}\phi^{a} + \tfrac{1}{2}\,\sigma\phi^{a}\phi^{a} - (3\sigma^{2}/2g), \tag{4}$$

where g is the coupling constant and the field σ is auxiliary. Its elimination produces the usual ϕ^4 interaction. The method of Refs. [15,16,19] elegantly exploits the properties of the d-dimensional Wilson-Fisher fixed point in that, for example, the (massless) propagators of the fields of Eq. (4) have simple power law behavior. In momentum space, representing the propagator by the same letter as the field, the leading asymptotic scaling forms in the critical region are

$$\phi(k) \sim \frac{A}{(k^2)^{\mu-\alpha}}, \quad \sigma(k) \sim \frac{B}{(k^2)^{\mu-\beta}}, \tag{5}$$

where $d = 2\mu$ and A and B are the momentum independent amplitudes that always appear in the combination $z = A^2B$ in the computation of the Feynman diagrams. The powers of the propagators are related to the usual critical exponents by

$$\alpha = \mu - 1 + \tfrac{1}{2}\eta, \quad \beta = 2 - \eta - \chi, \tag{6}$$

where χ is the anomalous dimension of the $\sigma\phi^2$ vertex. It can also be determined from a scaling law involving ν,

$$\chi = (1/\nu) - \eta - 2(\mu - 1), \tag{7}$$

where ν is proportional to the critical slope of the coupling constant of the O(N) nonlinear σ model that is in the same universality class as Eq. (4) in $2<d<4$. In the large N critical point technique the propagators in the Feynman diagrams of a Green's function are represented by Eq. (5). However, in their present form when they are used to compute $\eta_{\mathcal{O}}$ the leading order large N graphs diverge. To regularize these infinities the regulator Δ is introduced by setting $\chi \to \chi + \Delta$. Consequently the Feynman diagrams involve poles in Δ analogous to those in conventional perturbation theory where ϵ in $d = 4 - 2\epsilon$ is the (dimensional) regularization. It is the residues of these simple poles in Δ which are then used to extract $\eta_{\mathcal{O}}$. It is worth stressing that we will compute the exponent in d dimensions where d is arbitrary and ϵ is *not* used as a regularization.

For the large N renormalization of the composite operator T^{ab} it turns out that only those Feynman diagrams where the operator is not within a closed ϕ^a field loop will contribute to $\eta_{\mathcal{O}}$. This is a consequence of the traceless nature of the operator. Diagrams where T^{ab} is inside a closed ϕ^a loop vanish when one computes the group theory factor of the graph. Therefore, at leading order, $O(1/N)$, there is only one Feynman diagram to calculate and applying the method of Ref. [19], we find

$$\eta_{\mathcal{O},1} = -(\mu\eta_1)/(\mu - 2), \tag{8}$$

TABLE I. Values of crossover critical exponents from Ref. [9].

N	ϕ_c	β_c	γ_c
2	1.184(12)	0.830(12)	0.354(25)
3	1.271(21)	0.863(21)	0.41(4)
4	1.35(4)	0.90(4)	0.45(8)
5	1.40(4)	0.90(4)	0.50(8)
8	1.55(4)	0.94(4)	0.61(8)
16	1.75(6)	0.98(6)	0.77(12)

where

$$\eta_{\mathcal{O}} = \sum_{i=1}^{\infty} (\eta_{\mathcal{O},i}/N^i) \tag{9}$$

with

$$\eta_1 = -\frac{4\Gamma(2\mu-2)}{\Gamma(\mu+1)\Gamma(\mu-1)\Gamma(\mu-2)\Gamma(2-\mu)} \tag{10}$$

and

$$\eta = \sum_{i=1}^{\infty} (\eta_i/N^i). \tag{11}$$

Consequently, we have

$$\phi_c = \frac{1}{(\mu-1)} + \frac{2\mu\eta_1}{(\mu-1)(\mu-2)N} + O(1/N^2), \tag{12}$$

which is in *exact* agreement with Refs. [17,18], though extracted with a minimal amount of effort. In three dimensions, Eq. (12) gives

$$\phi_c = 2 - \frac{32}{\pi^2 N} + O(1/N^2) \tag{13}$$

or using a Padé approximant

$$\phi_c = \frac{2}{[1 + (16/\pi^2 N)]}. \tag{14}$$

Interestingly evaluating Eq. (14) for $N = 2$, 3, 5, and 16 we find the respective values $\phi_c = 1.105$, 1.298, 1.510, and 1.816. These are relatively close to the values obtained by other methods [9] that are given in Table I. Indeed the estimate for $N = 3$ is remarkably good. By contrast the direct evaluation of Eq. (13) gives respectively 0.379, 0.919, 1.352, and 1.797 indicating its poor convergence for low N.

To determine $\eta_{\mathcal{O},2}$ we have repeated the method on the $O(1/N^2)$ diagrams. Due to the way the large N expansion orders this would ordinarily mean that graphs up to five loops would have to be calculated. However, when the group theory factor is computed only six diagrams remain with a nonzero coefficient. These are comprised of four two-loop and two three-loop graphs. As a check on our method of calculation we have redetermined ν_2 from the evaluation of the exponent χ_2 using the same computer program written in the symbolic manipulation language FORM [22]. The method of extracting χ_2 is the same as that for $\eta_{\mathcal{O},2}$ since the Feynman diagrams for the latter are equivalent to those for the former when the operator insertion is replaced by the $\sigma\phi^2$ vertex. Therefore, the result of our calculation is

$$\eta_{\mathcal{O},2}=\left[\left(2\mu+5+\frac{14}{(\mu-2)}+\frac{8}{(\mu-2)^2}\right)v'+2\mu+2+\frac{1}{(\mu-2)}-\frac{8}{(\mu-2)^2}-\frac{8}{(\mu-2)^3}+\frac{1}{2(\mu-1)}\right]\eta_1^2, \tag{15}$$

where

$$R_1=\psi'(\mu-1)-\psi'(1),$$
$$R_2=\psi'(2\mu-3)-\psi'(2-\mu)-\psi'(\mu-1)+\psi'(1),$$
$$R_3=\psi(2\mu-3)+\psi(2-\mu)-\psi(\mu-1)-\psi(1),$$
$$v'=\psi(2\mu-2)+\psi(2-\mu)-\psi(\mu-2)-\psi(2), \tag{16}$$

and $\psi(x)$ is defined by $\psi(x)=\Gamma'(x)/\Gamma(x)$ where $\Gamma(x)$ is the Euler gamma function. Consequently,

$$\phi_c=\frac{1}{(\mu-1)}+\frac{2\mu\eta_1}{(\mu-1)(\mu-2)N}+\left[\frac{3\mu^2(8\mu^2-21\mu+14)R_1}{2(\mu-1)(\mu-2)^3}-\frac{\mu^2(2\mu-3)^2}{(\mu-1)(\mu-2)^3}[R_3^2+R_2]+\frac{\mu(4\mu^3-14\mu^2+10\mu+1)}{(\mu-1)^2(\mu-2)^2}v'-2+\frac{6}{(\mu-2)}-\frac{41}{(\mu-2)^2}-\frac{4}{(\mu-2)^3}-\frac{15}{(\mu-1)}+\frac{1}{(\mu-1)^2}+\frac{3}{2(\mu-1)^3}\right]\frac{\eta_1^2}{N^2}+O\left(\frac{1}{N^3}\right). \tag{17}$$

To check the correctness of Eq. (17) we have evaluated ϕ_c at $O(\epsilon^4)$ in $d=4-2\epsilon$ and compared with the previous dimensionally regularized four-loop $\overline{\text{MS}}$ perturbative calculation of the same critical exponent. The result (17) is in exact agreement, which is a nontrivial check on our computation since only three-loop graphs are present at $O(1/N^2)$. With Eq. (17) we can expand to a *new* order in ϵ and find

$$\phi_c=1+\epsilon+\epsilon^2+\epsilon^3+\epsilon^4+\epsilon^5-[8\epsilon-8\epsilon^3-16\{1-\zeta(3)\}\epsilon^4-24\{1-\zeta(4)\}\epsilon^5](1/N)+[64\epsilon-124\epsilon^2-4\{43+60\zeta(3)\}\epsilon^3+\{640\zeta(5)-360\zeta(4)+976\zeta(3)-155\}\epsilon^4+2\{800\zeta(6)-1840\zeta(5)+732\zeta(4)+128\zeta^2(3)-144\zeta(3)+61\}\epsilon^5](1/N^2)+O(\epsilon^6/N^3), \tag{18}$$

where $\zeta(n)$ is the Riemann zeta function and the order symbol represents independently higher-order terms in ϵ and $1/N$. The $O(\epsilon^5)$ coefficients will be important in future explicit five-loop $\overline{\text{MS}}$ perturbative calculations.

We are now in a position to examine the critical exponents in three dimensions. For the various ones we are interested in we have

$$\phi_c=2-\frac{32}{\pi^2N}-\frac{64[9\pi^2+16]}{9\pi^4N^2}+O\left(\frac{1}{N^3}\right),$$
$$\eta_c=\frac{32}{3\pi^2N}-\frac{512}{27\pi^4N^2}+O\left(\frac{1}{N^3}\right),$$
$$\eta_{\mathcal{O}}=\frac{8}{\pi^2N}+O\left(\frac{1}{N^3}\right). \tag{19}$$

For reference, the other intermediate exponents are

$$\eta=\frac{8}{3\pi^2N}-\frac{512}{27\pi^4N^2}+O\left(\frac{1}{N^3}\right),$$
$$\nu=1-\frac{32}{3\pi^2N}-\frac{32[27\pi^2+104]}{27\pi^4N^2}+O\left(\frac{1}{N^3}\right). \tag{20}$$

In addition we record that the values of two related crossover critical exponents are

$$\beta_c=1-\frac{32[\pi^2+8]}{\pi^4N^2}+O\left(\frac{1}{N^3}\right),$$
$$\gamma_c=1-\frac{32}{\pi^2N}-\frac{32[9\pi^2-40]}{9\pi^4N^2}+O\left(\frac{1}{N^3}\right), \tag{21}$$

which are defined through the hyperscaling laws

$$\beta_c=2\mu\nu-\phi_c\,,\gamma_c=2\phi_c-2\mu\nu. \tag{22}$$

Clearly the $O(1/N^2)$ correction to ϕ_c is large and the series appears to diverge. By contrast the $O(1/N^2)$ correction to $\eta_{\mathcal{O}}$ vanishes in three dimensions. We have repeated our earlier Padé approach for ϕ_c to see if the convergence is improved but this does not lead to a small change to the previous values for the exponents. This is in part due to the fact that the exponents η and ν do not lend themselves to improvement by this approach. Instead one way of gaining estimates from our large N results is to use the accepted values of η and ν and our value for $\eta_{\mathcal{O}}$. Indeed in Ref. [11] the four-loop estimate for ϕ_c was determined in an analogous fashion. Therefore, taking η to be 0.033 and 0.033 and ν to be 0.669 and 0.705 for $N=2$ and 3, respectively [7] we find the values for ϕ_c are 1.044 and 1.196. These are in poor agreement with the respective results of Ref. [9]. For the exponents β_c and γ_c the large N corrections are also large for small N and each series appears to converge slowly. To appreciate this we have evaluated the above expressions for larger values of N. For $N = 8$ we find $\phi_c = 1.475$, $\beta_c = 0.908$, and $\gamma_c = 0.567$. By contrast when $N = 16$ our expressions give 1.767, 0.977, and 0.790 for the same respective exponents which, by contrast, compare much more favorably with the respective values of 1.75(6), 0.98(6), and 0.77(12) of Ref. [9].

In order to improve the convergence of the series we have also examined the Padé-Borel resummation of the large N series. This involves determining the Borel function of the series that is defined by

$$\sum_{n=0}^{\infty} a_n x^n = \frac{1}{x} \int_0^{\infty} dt\ e^{-t/x} \sum_{n=0}^{\infty} \frac{a_n t^n}{n!}, \tag{23}$$

and then taking a Padé approximant of the integrand given that only several terms in the series are known. Therefore, for ϕ_c its Padé-Borel estimate is

$$\phi_c = 2N \int_0^{\infty} dt \frac{e^{-Nt}}{[1 - a_1 t + (a_1^2 - \frac{1}{2} a_2) t^2]}, \tag{24}$$

where

$$a_1 = -\frac{16}{\pi^2}, \quad a_2 = -\frac{32[9\pi^2 + 16]}{9\pi^4}. \tag{25}$$

We have evaluated the integral numerically for various values of N and recorded the results in Table II where the estimates for β_c and γ_c by the same method are also given. The final column is the sum of the estimates in the second and third columns and represents another way of estimating ϕ_c through the scaling relation since we have noted that the large N series for ϕ_c appears to diverge rapidly for low N. For $N \geqslant 4$ the large N estimates for ϕ_c and the sum $\beta_c + \gamma_c$ are in fairly reasonable agreement. For $N = 2$ and 3 the estimates undershoot those of Ref. [9] though the combination $\beta_c + \gamma_c$ is closer. For the other exponents the values for β_c are competitive for $N \geqslant 5$ whilst those for γ_c appear to be in good agreement for the lower range of N.

TABLE II. Padé-Borel estimates of crossover exponents.

N	ϕ_c	β_c	γ_c	$\beta_c + \gamma_c$
2	0.988	0.664	0.367	1.031
3	1.187	0.768	0.459	1.227
4	1.323	0.830	0.529	1.359
5	1.422	0.871	0.582	1.453
8	1.603	0.934	0.689	1.623
16	1.790	0.980	0.817	1.797

In conclusion, we have provided the $O(1/N^2)$ corrections to a set of crossover exponents related to the composite operator T^{ab} in O(N) ϕ^4 theory. Although the leading order exponents could be summed to give numerical estimates that are competitive with explicit perturbative calculations in three dimensions the *new* higher-order correction indicate that the series are slowly converging. Applying the Padé-Borel resummation technique generally improves the estimates in comparison with the results of Ref. [9] though it ought to be borne in mind that $O(1/N^2)$ results represent only three terms of a series in contrast to Ref. [9] which analyzed six terms of a series. Nevertheless since the critical exponents are computed in d dimensions they will complement future higher-order perturbative calculations and, further, the large N method can equally be applied to the determination of crossover exponents of bilinear and other composite operators to the same large N order in this and other scalar quantum field theories which underpin critical phenomena.

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